A Comparison Between Relaxation and Kurganov–Tadmor Schemes

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Summary. In this work we compare two semidiscrete schemes for the solution of hyperbolic conservation laws, namely the relaxation [JX95] and the Kurganov Tadmor central scheme [KT00]. We are particularly interested in their behavior under small time steps, in view of future applications to convection diffusion problems. The schemes are tested on two benchmark problems, with one space variable.

1 Motivation

We are interested in the solution of systems of equations of the form

$$u_t + f_x(u) = D\, p_{xx}(u), \tag{1}$$

where $f(u)$ is hyperbolic, i.e., the Jacobian of f is provided with real eigenvalues and a basis of eigenvectors for each u, while $p(u)$ is a nondecreasing Lipschitz continuous function, with Lipschitz constant μ and $D \geq 0$.

We continue the study of convection diffusion equations with the aid of high order relaxation schemes started in [CNPS06] for the case of the purely parabolic problem.

In many applications, such as multiphase flows in porous media, $p(u)$ is nonlinear and possibly degenerate. In these conditions, an implicit solution of the diffusion term can be computationally very expensive: in fact it may be necessary to solve large nonlinear algebraic systems of equations which, moreover, can be singular at degenerate points, i.e., where $p(u) = 0$. For this reason, it is of interest to consider the *explicit* solution of (1). This in turn poses one more difficulty. An explicit solution of (1) requires a parabolic CFL condition, that is, stability will restrict the possible choice of the time step Δt to $\Delta t \leq C(\Delta x)^2$, where Δx is the grid spacing. In other words, it may be necessary to choose very small time steps. But conventional solvers for convective operators typically work at their best for time steps close to a *convective* CFL, i.e., $\Delta t \leq C\Delta x$. When the time step is much smaller, they exhibit a very large artificial diffusion of the form $O((\Delta x)^{2r}/\Delta t)$, where r is the accuracy of the scheme, see for instance [KT00]. Clearly in these conditions artificial diffusion becomes very large for $\Delta t \to 0$.

As a first step to the numerical solution of problem (1), we concentrate on semidiscrete schemes for the solution of the convective part of (1). Such schemes enjoy an artificial diffusion which depends weakly on Δt, and are therefore particularly suited for the solution of convection–diffusion equations.

We will compare two semidiscrete methods for the integration of systems of hyperbolic equations. We are interested in the representation of solutions which can be characterized by strong gradients, and in the degenerate case, even by discontinuities. Moreover, we are interested in comparing the behavior of the schemes for small values of Δt, and for such small values of the time step, we will investigate the resolution of discontinuous solutions and the behavior of the error in a few test problems.

The schemes analyzed in this work are the Kurganov Tadmor central scheme proposed in [KT00], and the relaxation scheme proposed in [JX95]. These methods discretize the equations starting from very different ideas; however, they share some interesting properties. First of all, they are both semidiscrete schemes. Therefore, they require separate discretizations in space and time, which is the key to the fact that artificial diffusion depends mainly on space discretization. Secondly, they are both Riemann solver free methods. The Kurganov–Tadmor scheme is based on a central approach: the solution of the Riemann problem is computed on a staggered cell, before being averaged back on the standard grid. In this fashion, the numerical solution is updated on the edges of the staggered grid, where it is smooth, and can be computed via a Taylor expansion, with no need to solve the actual Riemann problem. The relaxation scheme instead moves the nonlinearities of the convective term to a stiff source term, and the transport part of the system becomes linear, with a fixed and well known characteristic structure. Thus again there is no need to use approximate or exact Riemann solvers.

For these reasons both schemes can be applied as black-box methods to a fairly general class of balance laws.

2 Results

For the Kurganov–Tadmor (KT) scheme we have followed the componentwise implementation of the method described in [KT00]. The scheme is written in conservation form, with numerical flux

$$\begin{aligned} F_{j+1/2}(t) = \tfrac{1}{2} \Big[& f(u^+_{j+1/2}(t)) + f(u^-_{j+1/2}(t)) \\ & - a_{j+1/2}(t) \left(u^+_{j+1/2}(t) - u^-_{j+1/2}(t) \right) \Big] , \end{aligned} \tag{2}$$

where $u^+_{j+1/2}(t)$ and $u^-_{j+1/2}(t)$ are the boundary extrapolated data, computed at the edges of each cell with a piecewise linear reconstruction at time t, and

$a_{j+1/2}(t)$ is a measure of the maximum propagation speed at the cell edge. For the case of systems of equations, in particular in the nonconvex case, this value must be carefully tuned, and it is the same for all components, when the scheme is implemented componentwise.

On the other hand, the relaxation scheme requires an accurate choice of the subcharacteristic velocities A^2. The relaxation system is

$$\begin{cases} \dfrac{\partial \mathbf{u}}{\partial t} + \dfrac{\partial \mathbf{v}}{\partial x} = 0 \\ \dfrac{\partial \mathbf{v}}{\partial t} + A^2 \dfrac{\partial \mathbf{u}}{\partial x} = -\dfrac{1}{\varepsilon} \left(\mathbf{v} - f(\mathbf{u}) \right) . \end{cases} \tag{3}$$

As $\varepsilon \to 0$, the system (3) formally relaxes to the original conservation laws, provided the subcharacteristic condition holds, namely that $(A^2 - (f'(u))^2)$ is positive-definite.

For a scalar conservation law, we take $A^2 = \max(|f'(u)|)$ as in [JX95], while for the Euler system of gas-dynamics we take A^2 to be the diagonal matrix with entries $\max_j(|u_j - c_j|)$, $\max_j(|u_j|)$, and $\max_j(|u_j + c_j|)$. Here u is the velocity and c is the speed of sound. We update these quantities at each time step, so that A^2 can be chosen as small as possible (in the paper [JX95] A^2 was chosen as a constant diagonal matrix but this results in a larger numerical diffusion).

Because of the diagonal form of A^2, the convective operator is block diagonal with 2×2 blocks. Each block is independently diagonalized and we compute the numerical fluxes using a second order ENO reconstruction [HEOC87].

We use the second order Heun Runge–Kutta method for the time integration of both the KT and the relaxation schemes.

Table 1 shows the errors in the L^1 norm for the linear advection equation $u_t + u_x = 0$ with initial data $u(x, 0) = \sin(2\pi x)$. We use the standard convective CFL condition $\Delta t = C\Delta x$ and the parabolic CFL, $\Delta t = C(\Delta x)^2$. We note that the errors are almost the same for the two schemes for the convective CFL, while the relaxation scheme seems superior for the parabolic CFL.

Table 1. Linear advection of a sine function

	Convective CFL		Parabolic CFL	
	KT	Relax	KT	Relax
20	2.03E−1	2.16E−1	6.19E−1	1.02E−1
40	7.58E−2	7.66E−2	2.04E−1	4.58E−2
80	2.71E−2	2.73E−2	9.10E−2	1.34E−2
160	8.22E−3	8.25E−3	2.67E−2	3.82E−3
320	2.29E−3	2.29E−3	7.62E−3	1.03E−3
640	6.11E−4	6.12E−4	2.06E−3	2.77E−4
1280	1.61E–4	1.61E–4		

Errors in L^1 at $t = 1$.

A key requirement for a numerical scheme for conservation laws is the ability to pick the entropy solution in nonconvex problems. Here we show a Riemann problem for the nonconvex flux $f(u) = (u^2 - 1)(u^2 - 4)/4$, as in [KT00]. The Riemann problem breaks into two shocks connected by a rarefaction wave. The results are shown in Fig. 1. Clearly both schemes are able to resolve the correct discontinuities and they have approximately the same resolution, the KT scheme being slightly less diffusive.

Figure 2 shows the density component of the Lax Riemann problem in gas dynamics. The accurate choice suggested above for the matrix A^2 in the relaxation system yields a slightly higher resolution than KT.

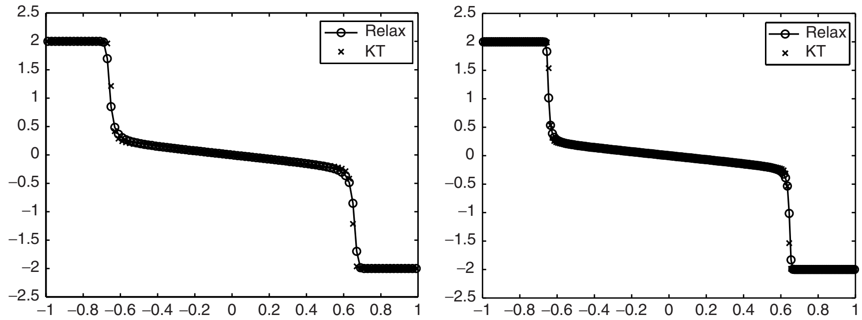

Fig. 1. Nonconvex flux. Kurganov–Tadmor and relaxation schemes, with $n = 100$ (*left*) and $n = 200$ (*right*)

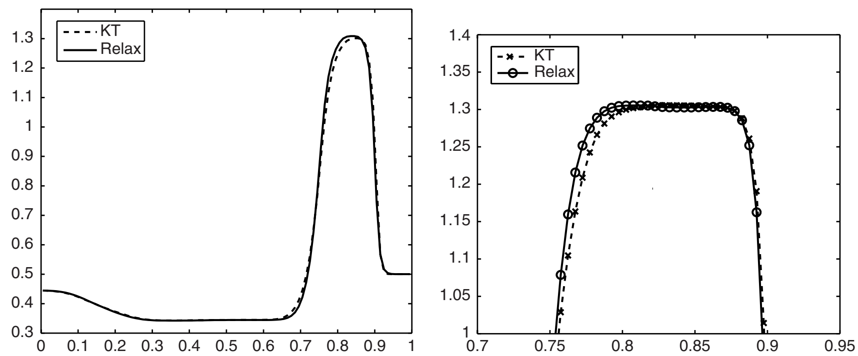

Fig. 2. Lax Riemann problem, density component. KT (*dashed*) and relaxation (*solid line*) with $n = 100$ (*left*) and $n = 200$ (*right*), where a detail of the density peak is shown

3 Concluding Remarks

We have compared two semidiscrete schemes for conservation laws. We find that although the schemes are constructed with very different philosophies, they yield comparable results on some significant test problems. We think that the relaxation scheme is slightly more robust, since it results from the relaxation of a viscous profile, provided the subcharacteristic condition is satisfied. Also, the actual errors obtained with a parabolic CFL in Table 1 seem to favor the relaxation scheme.

We also wish to mention higher order extensions of the schemes studied in this work: namely the third order central upwind scheme described in [KNP01], endowed with a more carefully crafted artificial diffusion with respect to [KT00] and the third order extension of the relaxation scheme proposed in [Sea06].

References

[CNPS06] F. Cavalli, G. Naldi, G. Puppo, and M. Semplice. High order relaxation schemes for non linear diffusion problems. 2006. Submitted to SINUM.

[HEOC87] Ami Harten, Björn Engquist, Stanley Osher, and Sukumar R. Chakravarthy. Uniformly high-order accurate essentially nonoscillatory schemes. III. *J. Comput. Phys.*, 71(2):231–303, 1987.

[JX95] S. Jin and Z. Xin. The relaxation schemes for systems of conservation laws in arbitrary space dimensions. *Comm. Pure and Appl. Math.*, 48:235–276, 1995.

[KNP01] Alexander Kurganov, Sebastian Noelle, and Guergana Petrova. Semidiscrete central-upwind schemes for hyperbolic conservation laws and Hamilton-Jacobi equations. *SIAM J. Sci. Comput.*, 23(3):707–740 (electronic), 2001.

[KT00] Alexander Kurganov and Eitan Tadmor. New high-resolution central schemes for nonlinear conservation laws and convection-diffusion equations. *J. Comput. Phys.*, 160(1):241–282, 2000.

[Sea06] Mohammed Sead. High-resolution relaxation scheme for the two-dimensional Riemann problems in gas dynamics. *Numer. Methods Partial Differential Equations*, 22(2):397–413, 2006.